On the renormalisability of gauge invariant extensions of the squared gauge potential

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Abstract

We show that gauge invariant extensions of the local functional $\mathcal{O} = \frac{1}{2}\int d^4x A^2$ have long range non localities which can only be "renormalised" with reference to a specific gauge. Consequently, there is no gauge independent way of claiming the perturbative renormalisability of these extensions. In particular, they are not renormalisable in the modern sense of Weinberg and Gomis. Critically, our study does not support the view that ghost fields play an indispensable role in the extension of a local operator into a non-local one as claimed recently in the literature.

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1 Introduction and summary

In recent years there has been an increasing interest in the calculation of the square of the gauge potential. The idea that condensates in Yang-Mills encode non-perturbative effects has a long history. In particular, a condensate of A^2 was considered some years ago [1] but due to its gauge dependence it has not been the focus of much investigation. However, it has been argued in [2] that a non-local gauge invariant functional associated to A^2 contains information on topological structures of the Yang-Mills vacuum which is revealed by a non vanishing expectation value of the operator. This scenario is realised in the compact $U(1)$ gauge theory [3] where magnetic monopoles condense [2] for large coupling.

The prospect that A^2 might indeed carry physically relevant information has motivated different groups to calculate the expectation value of the local operator $\mathcal{O} = \frac{1}{2}\int d^4x(A_\mu^a)^2$ in covariant gauges. There have been both analytic [4, 5] and numerical studies in the lattice [6]. The working model used has been $SU(2)$ Yang-Mills. The underlying idea is that the presumed non-local gauge invariant operator associated with A^2 takes a local form in the particular gauge where the calculations are carried out. In many cases, what are actually considered are extensions of $\mathcal{O}$ involving ghost fields [4]. The introduction of the ghost fields makes it possible to write the local gauge fixed operator as a gauge fixed BRS invariant operator. The expected advantages of this type of approach are the benefit of using the BRS invariance to guarantee the renormalisability of the composite operator and the prospect that due to BRST cohomology theorems an observable is associated to the BRS invariant operator. The latter expectation has been shown not to be fulfilled [7], more about which will be discussed below. The question of renormalisability will be addressed in this letter.

The renormalisability of a non local gauge invariant extension of a local operator (which is given by a local gauge non-invariant expression in a specific fixed gauge) is addressed here. To summarise our main results, we present strong arguments that the methods being used to evaluate the expectation value of A^2, or its local extensions, *require a renormalisation procedure that is slave to a particular gauge fixing*. In other words, we can not extend the standard renormalisation procedure, based in the introduction of counterterms order by order in perturbation theory, outside a unique fixed gauge. The particular gauge in question is the one where the non local extension is expressed by a local functional. When this occurs, there are no guarantees that the resulting renormalised expectation value of the non local gauge invariant extension is gauge independent. This kind of difficulty should not come as a surprise. In fact, it would be surprising if the perturbative methods on which the known renormalisation procedure is based could capture reliable information on the topological structure that is expected to be responsible for a non vanishing expectation value of the gauge invariant extension of A^2 [2].

In our study we illustrate the properties of gauge invariant extensions of local functionals. We aim at clarifying, via specific examples, the relation between a

functional which is local in a particular gauge (but not necessarily gauge invariant), and its gauge invariant extension (which is not necessarily local). We show that the non localities found are not perturbatively local because they can not be expressed in terms of an infinite derivative expansion. We believe that the implications of this observation have not been clearly emphasised in the literature, as attested by the absence of any debate about it in recent works. It is precisely these dangerous infrared modes that make it hard to define a gauge independent renormalisation for the gauge invariant extensions of local functionals. This observation supports the remark in [2] that the expectation value receives important contributions from both large and small distances. Our arguments on renormalisability are based on the notion of renormalisation in the modern sense [8] which relies on BRST cohomology theorems. The BRST terminology will therefore be frequently used here, even though it is not always necessary.

The expectation value of the extensions of A^2 can only be claimed to be "renormalisable" in the particular gauge where they have a local expression. If we try to define renormalisability in any gauge by going back to the gauge where the functional takes a local form, we become tied to this gauge and renormalisability is no longer a gauge independent property. Implicitly, this is what has been done to date in the literature [4, 5, 6]. We emphasise the distinction between this situation and, for example, the case of the standard model where renormalisability can be guaranteed from gauge to gauge.

The non locality of a gauge invariant extension of $\mathcal{O}$ is unavoidable if the operator is to be associated to an observable. In a recent analysis [7] we have studied the operator $\mathcal{O}$ in the powerful context of the antifield Batalin-Vilkovisky formalism [9] using local BRST extension and deformation techniques [10]. By analysing $\mathcal{O}$ with a ghost sector we have shown that ghost condensates are an artifact of gauge fixed actions. A by-product of this analysis was the observation that there is no local observable associated with an on-shell BRS invariant mass dimension two local functional in $SU(N)$ Yang-Mills theories. This observation has important consequences.

On the one hand, it illustrates that gauge-fixed BRS invariance and gauge invariance [11] are not always equivalent. In this respect it is important to realise that the relation between classical observables and gauge-fixed BRST cohomology is not a straightforward one, unlike the case for the gauge independent BRST cohomology. Only in this last case are we guaranteed the existence of a one-to-one correspondence between classical observables and elements of the BRST cohomology at zero ghost number for both local and non-local functionals. For *non-local* functionals the correspondence is one-to-one even for the gauge-fixed case, while for local functionals extra conditions (see discussion at end of Sect. 6) which are not fulfilled by $\mathcal{O}$ are required [11].

On the other hand, it indicates that the only viable extensions of $\mathcal{O}$ ought to be non local. The possibility of associating a non-local observable to a mass dimension two operator has since been exploited by Kondo [12] in order to argue for a physical

meaning to $\mathcal{O}$ in the Abelian gauge theory. In his study the author makes use of gauge-fixed cohomology to derive the possible physical interpretation of $\mathcal{O}$. Some subtle aspects of Kondo's arguments require re-evaluation. In general, his line of reasoning would make any on-shell BRS invariant operator a candidate for a physical observable. If we have in mind that one of the features of gauge theories is their constrained structure this is too generic.

We state here three key observations that conflict with Kondo's view. Firstly, the vacuum expectation value of $\mathcal{O}$ is only equivalent to that of its gauge invariant extension in a particular gauge and, therefore, this equivalence cannot be the origin for gauge independent statements. Secondly, ghost fields are not essential to extend a local gauge variant operator into a non local invariant one. Thirdly, the way a gauge condition is implemented has an effect on the BRST transformations. All these issues will be discussed throughout this letter along side the central question of renormalisation.

To avoid the technical difficulties involved in Yang-Mills theories the explicit examples we use in this letter are in Maxwell's theory. However, all the properties of gauge invariant extensions we illustrate are generic, and also apply to the non-Abelian case. In Sect. 2 we present a general discussion of gauge invariant extensions. This is followed by an explicit study of non-local extensions of $\mathcal{O}$ in Maxwell's theory for arbitrary linear gauges in Sect. 3. We illustrate that the properties of gauge invariant extensions are related to those of the gauge in which $\mathcal{O}$ is initially specified, and emphasise the fact that locality is often not preserved by the extension procedure. In Sect. 4 we discuss the two standard ways of implementing gauge fixing because of the importance of specifying the gauge from which the extension is constructed. In Sect. 5 we analyse the gauge dependent nature of the relation between the expectation values of $\mathcal{O}$ and its gauge invariant extension. Various subtle issues concerning the renormalisation of non-local gauge invariant extensions are discussed in Sect. 6. In Sect. 7 we present a final discussion on our analysis.

2 Gauge invariant extension

Consider a local functional $\mathcal{O}$ and a fixed gauge. The latter is specified by the gauge fixing fermion Ψ following the prescription where the gauge fixing plus ghost sector of the action is $\int s\Psi$, with s the BRST operator. The functional $\mathcal{O}$ can always be extended off the gauge Ψ in a gauge invariant way. The resulting functional, the *gauge invariant extension*, which we denote by $\mathcal{O}^{\uparrow\Psi}$, is by construction strongly gauge invariant [13]. Unless $\mathcal{O}$ is itself gauge invariant the relation between $\mathcal{O}$ and $\mathcal{O}^{\uparrow\Psi}$ depends on the specified gauge Ψ, therefore we keep Ψ as an upper script in the extension as a reminder. An important gauge dependent identity that follows from the construction of the extension $\mathcal{O}^{\uparrow\Psi}$ is the equality

$$\langle\mathcal{O}\rangle_\Psi = \langle\mathcal{O}^{\uparrow\Psi}\rangle_\Psi, \tag{1}$$

where $\langle \cdot \rangle_\Psi$ denotes the expectation value evaluated in the specific gauge Ψ. We will return to (1) at a later stage.

Note that though the gauge invariant extension is not necessarily unique for a given functional $\mathcal{O}$ in a gauge Ψ, two different extensions $\mathcal{O}^{\uparrow\Psi}$ and $\mathcal{O}'^{\uparrow\Psi}$ always have the same expectation value. This follows from the fact that the ambiguity is proportional to terms that vanish modulo the equations of motion of the *gauge invariant* action or s-exact terms.

In the examples discussed in this letter the extensions can be computed using only fields and ghosts. Moreover, the final explicit expressions for the gauge invariant extensions can be written without ghosts. Therefore, contrary to [12] *we do not find evidence that ghost condensates are necessary to convert local operators into non-local ones.*

The gauge Ψ which we shall call the "*base gauge*" has an important role in determining which properties and symmetries of $\mathcal{O}$ are carried along to $\mathcal{O}^{\uparrow\Psi}$. For example, if we extend a covariant operator $\mathcal{O}$ from a non covariant base gauge, the resulting extension is not expected to be covariant. This will be illustrated below. Independently of the base gauge, another property of $\mathcal{O}$ that the gauge invariant extension does not normally preserve is locality. Indeed, the extension is in general non local unless $\mathcal{O}$ is local and gauge invariant modulo the equations of motion of the *gauge invariant* action.

The construction of a gauge invariant extensions of a functional starting from a base gauge is very generic and in this sense it is always possible to associate a gauge invariant quantity to any $\mathcal{O}$. It should however, be emphasised that the methods used here only apply for extensions on a local patch because it is still possible to have obstructions due to the topological structure of the configuration space. As long as one works in perturbation theory these obstructions are avoided.

3 The A_μ^2 functional in the Maxwell theory

Consider the free Abelian gauge theory in four dimensions and let $\mathcal{O}$ denote the gauge dependent mass dimension two local functional

$$\mathcal{O} = \frac{1}{2}\int d^4x\ A_\mu^2 \,. \tag{2}$$

From the BRS transformation of the gauge potential, $sA_\mu = \partial_\mu C$, we have that the variation of $\mathcal{O}$ is given by

$$s\mathcal{O} = \int d^4x\ A_\mu\, \partial^\mu C = -\int d^4x\ \partial{\cdot}A\ C. \tag{3}$$

It follows from (3) and the discussion in [7] that $\mathcal{O}$ can not be added to the action as a mass term without effectively changing the physical content of the theory. The functional $\mathcal{O}$ is used in this letter to construct gauge invariant extensions from

various base gauges. This will provide us with explicit examples to study some general properties of these extensions.

We start by computing the gauge invariant extension of $\mathcal{O}$ for a general linear gauge Ψ_ℓ as the gauge base. The gauge condition is given by

$$\ell\cdot A \equiv \ell^\mu A_\mu = 0\,, \tag{4}$$

where ℓ^μ is an A_μ independent linear operator. Three familiar choices will be considered here,

$$\ell^\mu = \partial^\mu, \;\text{ Lorentz gauge } (\partial\cdot A = 0), \tag{5}$$
$$\ell^\mu = n^\mu, \;\text{ general axial gauge } (n\cdot A = 0), \tag{6}$$
$$\ell^\mu = \partial^\mu - \delta_0^\mu \partial^0, \;\text{ Coulomb gauge } (\vec{\partial}\cdot\vec{A} = 0), \tag{7}$$

where n^μ is a fixed 4-vector. The idea behind the calculation of $\mathcal{O}^{\uparrow\Psi_\ell}$ is very simple. Consider the infinitesimal variation of $\mathcal{O}$ along the gauge orbit when the potential is shifted away from the base gauge. Then look at how to modify $\mathcal{O}$ so it is parallel transported along the gauge orbit. Here for later convenience we consider the variations of the potential to be of the form of a BRS transformation where a ghost field appears at the place of the infinitesimal variation of the gauge parameter.

By applying the linear operator ℓ_μ on both sides of $sA_\mu = \partial_\mu C$ we obtain a non-local expression for the ghost field in terms of the gauge potential,

$$C = s\left(\frac{\ell\cdot A}{\ell\cdot\partial}\right). \tag{8}$$

For example, in the Lorentz gauge, $\frac{\ell\cdot A}{\ell\cdot\partial} = \frac{\partial\cdot A}{\Box} = -\int d^4k\, \frac{k^\mu \tilde{A}_\mu(k)}{4\pi^2 k^2}\, e^{ikx} + h.c.$, in the usual representation using Fourier transforms and distribution theory.

Using (8) it is now straightforward to determine $\mathcal{O}^{\uparrow\Psi_\ell}$. From (3) and (8) we have

$$\begin{aligned} s\mathcal{O} &= -\int d^4x\; \partial\cdot A\; s\left(\frac{\ell\cdot A}{\ell\cdot\partial}\right) \\ &= -\int d^4x\, \left(s\left(\frac{\ell\cdot A}{\ell\cdot\partial}\;\partial\cdot A\right) - \frac{\ell\cdot A}{\ell\cdot\partial}\;\Box C\right) \\ &= -s\int d^4x \left(\frac{\ell\cdot A}{\ell\cdot\partial}\;\partial\cdot A - \frac{1}{2}\;\frac{\ell\cdot A}{\ell\cdot\partial}\;\Box\frac{\ell\cdot A}{\ell\cdot\partial}\right). \end{aligned} \tag{9}$$

and we arrive at the BRS invariant extension

$$\mathcal{O}^{\uparrow\Psi_\ell} = \frac{1}{2}\int d^4x \left(A^2 + 2\,\frac{\ell\cdot A}{\ell\cdot\partial}\;\partial\cdot A - \frac{\ell\cdot A}{\ell\cdot\partial}\;\Box\frac{\ell\cdot A}{\ell\cdot\partial}\right), \tag{10}$$

which is also strongly gauge invariant in any local patch. The functional in (10) can be naturally identified as the gauge invariant extension of $\mathcal{O}$ in a linear gauge in the sense that

$$\mathcal{O}^{\uparrow\Psi_\ell}\Big|_{\ell\cdot A=0} = \mathcal{O}\,. \tag{11}$$

We can see from (10) that the extension depends on the base gauge has expected. In particular, for the Lorentz gauge we have

$$\mathcal{O}^{\uparrow\Psi_{\rm L}} = \frac{1}{2}\int d^4x\left(A^2 + \frac{\partial\cdot A}{\Box}\,\partial\cdot A\right), \tag{12}$$

which is clearly non local though $\mathcal{O}$ is local. We make the important observation that the non locality in (12) can not be expanded in a Taylor series. This property will be central to the discussion in Sect. 6. Similar types of non locality occur for extensions from other base gauges. For the axial gauge we have

$$\mathcal{O}^{\uparrow\Psi_{\rm A}} = \frac{1}{2}\int d^4x\left(A^2 + 2\,\frac{n\cdot A}{n\cdot\partial}\,\partial\cdot A - \frac{n\cdot A}{n\cdot\partial}\,\Box\,\frac{n\cdot A}{n\cdot\partial}\right), \tag{13}$$

and for the Coulomb gauge

$$\mathcal{O}^{\uparrow\Psi_{\rm C}} = \frac{1}{2}\int d^4x\left(A^2 + 2\,\frac{\vec{\partial}\cdot\vec{A}}{\vec{\partial}^{\,2}}\,\partial\cdot A - \frac{\vec{\partial}\cdot\vec{A}}{\vec{\partial}^{\,2}}\,\Box\,\frac{\vec{\partial}\cdot\vec{A}}{\vec{\partial}^{\,2}}\right). \tag{14}$$

Another property of the extensions concerns the effect the base gauge has upon the symmetries of $\mathcal{O}$. In the above examples we always started with a covariant operator but only the extension (12) preserves covariance. The covariance in (13) and (14) is lost in the process of extending $\mathcal{O}$ away from a non covariant base gauge.

4 Gauge fixing implementation

By definition the gauge invariance extension requires the choice of a specific base gauge as a starting point. It is therefore interesting to analyse how the extension might be affected by the gauge fixing procedure. When BRS techniques are used there are two standard implementations to fix the gauge, the delta function and the Gaussian average.

So far we have implemented the gauge fixing by requiring a gauge condition to be explicitly satisfied, (4). In a path integral representation this corresponds to implementing the gauge condition via a delta function. As an example, for the Lorentz gauge, (5), the gauge fermion is $\Psi_{\rm L}^{(\delta)} = \bar{C}\,(\partial\cdot A)$. The corresponding gauge fixing and ghost sector of the action is $\int s\Psi_{\rm L}^{(\delta)} = \int b\,(\partial\cdot A) - \bar{C}\,\Box\,C$, where b is the auxiliary Nakanishi-Laudrup scalar. It follows

$$\int \mathcal{D}[A_\mu,\bar{C},C,b]\exp i(S + {\textstyle\int} s\Psi_{\rm L}^{(\delta)}) = \int \mathcal{D}[A_\mu]\,\det\Box\;\delta(\partial\cdot A)\exp iS. \tag{15}$$

We consider now the other common way of implementing gauge fixing: Gaussian averaging of the gauge condition. This implementation is equivalent to the delta function one at the level of the gauge independent BRST antifield formalism. However, the Gaussian averaging is the appropriate one to introduce the gauge-fixed

BRST cohomology and analyse its relation to the off-shell gauge invariant formulation [11]. As we will see, this implementation is more general as it contains the previous in a specified limit.

The gauge fermion that implements the Lorentz condition by Gaussian averaging is $\Psi_{\rm L}^{(\rm Gauss)} = \bar{C}\,(\partial\cdot A - \frac{\alpha}{2}\,b)$, where α is the gauge fixing parameter. It follows from the gauge-fixed action that the equation of motion for the auxiliary field is $b = \frac{1}{\alpha}\,\partial\cdot A$. The on-shell gauge-fixed BRS transformations are obtained after integrating over b i.e. by implementing the b equation of motion. As an example, in the Lorentz gauge the off-shell BRS transformations $s\bar{C} = b$, $sb = 0$ become $s\bar{C} = \frac{1}{\alpha}\,\partial\cdot A$, $sb = 0$. For a delta function implementation of a gauge fixing condition these on-shell transformations can not be derived because b only enters linearly in the gauge-fixed action.

The path integral representation for the Gaussian averaging of the Lorentz condition is

$$\int \mathcal{D}[A_\mu, \bar{C}, C, b] \exp i(S + \textstyle\int s\Psi_{\rm L}^{(\rm Gauss)}) =$$

$$\int \mathcal{D}[A_\mu] \,\det\Box\;\; \exp i(S + \tfrac{1}{2\alpha} \textstyle\int (\partial\cdot A)^2). \tag{16}$$

In the limit $\alpha \to 0$ the delta function implementation is recovered.

5 The expectation value of $\mathcal{O}^{\uparrow\Psi_\ell}$

Let us consider for the moment the Gaussian average gauge fixing implementation. The phase space needs to be extended to include the antighost and the auxiliary field and the general linear gauge corresponds to the condition $\alpha b = \ell\cdot A$. Then, following an analogous approach to the one of Sect. 3, the gauge invariant extension of $\mathcal{O}$ for a base gauge specified by this condition is

$$\mathcal{O}^{\uparrow\Psi_\ell} = \frac{1}{2}\int d^4x \left(A^2 + 2\,\frac{\ell(A,b;\alpha)}{\ell\cdot\partial}\;\partial\cdot A - \frac{\ell(A,b;\alpha)}{\ell\cdot\partial}\;\Box\frac{\ell(A,b;\alpha)}{\ell\cdot\partial}\right), \tag{17}$$

with $\ell(A,b;\alpha) = \ell\cdot A - \alpha b$. Note that the $\alpha = 0$ choice corresponds to the extension (10). Moreover, the right-hand sides of (10) and (17) are equal up to s-exact terms as they only differ by terms involving the auxiliary field. They correspond therefore to the same gauge invariant functionals and we will be using the simpler form (10). At this point, it is also important to note that if we modify the integrand of the functional by adding a ghost sector, $\frac{1}{2}A^2 \to \frac{1}{2}A^2 - \alpha C\bar{C}$, and compute the gauge invariant extension, the resulting extension will only differ by an s-exact term, $-\alpha s\int \bar{C}\,(\ell\cdot\partial)^{-1}\ell(A,b;\alpha)$. It should be remarked that this is not a specific property for this functional $\mathcal{O}$, as it follows alone from the fact that the non-local gauge invariant extension of any term involving $C\bar{C}$, or for this purpose any other auxiliary fields, will always give trivial elements on the cohomology of s. Moreover, as it has been shown in [7] $C\bar{C}$ does not have local extensions.

For a general linear gauge, $\Psi_\ell = \ell\cdot A - \frac{1}{2}\alpha b$, the equation of motion of b reduces to

$$b = \frac{1}{\alpha}\,\ell\cdot A\,, \tag{18}$$

and therefore $\ell(A, b; \alpha) = 0$. From (18) it follows that the on-shell gauge-fixed BRS symmetry in the linear gauge is expressed by

$$s_{\Psi_\ell}\bar{C} = \frac{1}{\alpha}\,\ell\cdot A\,, \tag{19}$$

where s_{Ψ_ℓ} is the corresponding gauge-fixed BRS operator. At this level the equation of motion has been already implemented or equivalently, the b field has been integrated over. By taking (19) into account, the non-local terms on the right-hand side of (10) can be expressed as

$$\frac{\ell\cdot A}{\ell\cdot\partial}\,\partial\cdot A = \alpha\bar{C}\frac{\Box}{\ell\cdot\partial}\,C + s_{\Psi_\ell}\Big(\frac{\alpha\bar{C}}{\ell\cdot\partial}\,\partial\cdot A\Big)\,, \tag{20}$$

and

$$\frac{\ell\cdot A}{\ell\cdot\partial}\,\Box\frac{\ell\cdot A}{\ell\cdot\partial} = \alpha\bar{C}\frac{\Box}{\ell\cdot\partial}\,C + s_{\Psi_\ell}\Big(\frac{\alpha\bar{C}}{\ell\cdot\partial}\,\Box\frac{\ell\cdot A}{\ell\cdot\partial}\Big)\,. \tag{21}$$

Inserting (20-21) into (10) gives the explicit relation between $\mathcal{O}$ and $\mathcal{O}^{\uparrow\Psi_\ell}$,

$$\mathcal{O}^{\uparrow\Psi_\ell} = \mathcal{O} + \frac{\alpha}{2}\int d^4x\;\bar{C}\frac{\Box}{\ell\cdot\partial}\,C + s_{\Psi_\ell}\mathcal{B}\,, \tag{22}$$

where $\mathcal{B} = \alpha\int d^4x\frac{\bar{C}}{\ell\cdot\partial}(\partial\cdot A - \frac{1}{2}\Box\frac{\ell\cdot A}{\ell\cdot\partial})$ is a functional of the fields and ghosts. If we use (17) instead of (10) as the expression for the gauge invariant extension, the relation (22) remains valid but $\mathcal{B}$ is different. From this equation the expectation value of $\mathcal{O}$ is in general not equal to $\mathcal{O}^{\uparrow\Psi_\ell}$ and an equality is only guaranteed in the gauge Ψ_ℓ. Therefore, no gauge independent statement can be made between $\langle\mathcal{O}^{\uparrow\Psi_\ell}\rangle$ and $\langle\mathcal{O}\rangle$.

In order to clarify this point we take a closer look at the last two terms in the right-hand side of (22). Let us first recall the standard principle behind Ward identities. Consider δ to denote a classical symmetry of the action. Then for any functional $\mathcal{F}$ we have $\langle\delta\mathcal{F}\rangle = 0$. As far as s_{Ψ_ℓ} is concerned, as this refers solely to a symmetry of the gauge-fixed action for $\Psi = \Psi_\ell$ we can only state that $\langle s_{\Psi_\ell}\mathcal{F}\rangle_{\Psi_\ell} = 0$. Therefore, $\langle s_{\Psi_\ell}\mathcal{B}\rangle_{\Psi_\ell} = 0$, but in general $\langle s_{\Psi_\ell}\mathcal{B}\rangle_{\Psi} \neq 0$.

Next, consider the identity

$$s_{\Psi_\ell}\Big(\bar{C}\,\frac{\partial\cdot A}{\ell\cdot\partial}\Big) = \frac{1}{\alpha}\,\ell\cdot A\,\frac{\partial\cdot A}{\ell\cdot\partial} - \bar{C}\frac{\Box}{\ell\cdot\partial}C\,. \tag{23}$$

The expectation value of the left-hand side vanishes in the gauge Ψ_ℓ. The same also applies to the first term on the right-hand side. This follows from the off-shell identity $\langle b\,\frac{\partial\cdot A}{\ell\cdot\partial}\rangle = \langle s_{\rm aux}(\bar{C}\,\frac{\partial\cdot A}{\ell\cdot\partial})\rangle = 0$, where $s_{\rm aux} = s$ when acting on $\bar{C}$ and b

and gives zero on the other fields. Because the b field only enters linearly in this identity $\langle \ell \cdot A \frac{\partial \cdot A}{\ell \cdot \partial} \rangle_{\Psi_\ell} = 0$. Therefore, we have from (23) that in the general linear gauge $\langle \bar{C} \frac{\Box}{\ell \cdot \partial} C \rangle_{\Psi_\ell} = 0$ for the Maxwell theory. We then arrive at the *de facto* gauge dependent equality

$$\langle \mathcal{O} \rangle_{\Psi_\ell} = \langle \mathcal{O}^{\uparrow \Psi_\ell} \rangle_{\Psi_\ell} \tag{24}$$

as expected from (1) which is a direct consequence of the construction of gauge invariant extensions.

6 On the renormalisation of non-local functionals

In this section we discuss the perturbative renormalisability of the operator $\mathcal{O}$ in the *modern sense* as introduced by Gomis and Weinberg [8]. This criterion extends the Dyson one by allowing terms that are not power counting renormalisable. A theory is said to be renormalisable in the modern sense, if the symmetries of the bare action provide constraints that are sufficient to eliminate all the infinities. The symmetries of the bare action are encoded in the BRST symmetry of the gauge invariant action in the antifield formalism which is gauge independent. Gauge independent statements on the renormalisability of a given gauge theory are made possible by the close link between this renormalisation criterion and the cohomology of the BRST transformations generated by the action. Well established local BRST cohomology theorems [14, 15] provide the criteria to identify all the possible local counterterms. Contrary to the power-counting renormalisation criterion, there is no limit on the mass dimension of the allowed terms in the bare action. Therefore, an infinite number of counterterms are viable.

A sufficient condition for the renormalisability of the theory is the existence of an independent coupling in the action for each non-trivial element of the BRST cohomology. It is important to note that we can add any local term to the action compatible with the theory symmetries. In particular, we can add a non-local term in the form of an infinite number of derivative terms. It is still possible in this case to have a theory that is renormalisable in the modern sense because each derivative term is local, as required by the Quantum Action Principle [16]. An example occurs when the non locality enters through terms of the form $(\Box + m^2)^{-1}$ which can be expressed as an infinite sum of local terms $\sum_{n=0}^{\infty} m^{-(2n+2)}(-\Box)^n$, as long as $m \neq 0$. In this sense, even the Wilson loop is a perturbatively local quantity because it can be expressed in terms of an infinite series of local terms [17].

Here we are interested, in particular, in the renormalisability of a non-local gauge invariant functional like the extensions (12-14). The non locality in these extensions can not be expressed in terms of an infinite series of local terms. From the discussion in the previous paragraph we conclude that there is no gauge independent way in which $\mathcal{O}^{\uparrow \Psi}$ is renormalisable in the modern sense. Because of the formal relation $\langle \mathcal{O}^{\uparrow \Psi} \rangle = -i \frac{\delta}{\delta J} \int \mathcal{D}\phi \exp(iS[\phi] + iJ\mathcal{O}^{\uparrow \Psi})|_{J=0}$, as far as the role of the non locality

is concerned, the non renormalisability of $\mathcal{O}^{\uparrow\Psi}$ can be inferred from that of theory where the functional $\mathcal{O}$ is coupled to a source J and inserted to the action.

This, of course, is not in contradiction with the fact that an extension $\mathcal{O}^{\uparrow\Psi}$ can be perturbatively renormalisable in the base gauge where it takes a local form. What happens in this particular case is that *the "local" counterterms that make the functional renormalisable in this gauge can not be expressed as a series of local terms in other gauges.* We used "local" in the last sentence, to emphasise that for the consistency of the renormalisation procedure, locality should not be restricted to a particular gauge. However, this is not guaranteed in the present examples and therefore the Quantum Action Principle, which requires all the counterterm to be local is not ensured for other gauges.

It is interesting to see that starting from a gauge where $\mathcal{O}$ is multiplicatively renormalisable, the covariant gauge with $\alpha = 0$, that the renormalisability can not be "extended" to $\mathcal{O}^{\uparrow\Psi}$ *without having to redefine the standard renormalisation procedure.* A local functional can be associated to a gauge invariant quantity if it fulfills the two following conditions [11]:

1. it must be on-shell BRS invariant;
2. it must not break the nilpotency of the BRS symmetry when it is added to the gauge-fixed action.

In order for these conditions to hold, one must use the Gaussian averaging implementation of the gauge fixing. With a delta function implementation the first condition is not even satisfied (for non gauge invariant functionals). For example, consider $\mathcal{O}$ in the gauge $\partial\cdot A = 0$. From (3) we see that the BRS variation of $\mathcal{O}$ can not vanish on-shell as the equations of motion in the gauge $\Psi_{\mathrm{L}}^{(\delta)}$ are $\partial^\mu F_{\mu\nu} + \partial_\nu b = \Box C = \Box\bar{C} = 0$. The only way to have $s\mathcal{O} = 0$ is to set $\partial\cdot A = 0$, i.e. impose the gauge condition "by hand". The subtle distinction on the implications of these different gauge fixing implementations was not clearly distinguished in [12].

The first condition can be satisfied by considering the modified operator $A_0 = \mathcal{O} - \alpha\int d^4x\, C\bar{C}$ in the gauge fixed by $\Psi_{\mathrm{L}}^{(\mathrm{Gauss})}$, where now $sA_0 = 0$ modulo the equation of motion of b, $b = \frac{1}{\alpha}\,\partial\cdot A$. However, this operator does not fulfill the second condition because $s_\Psi^2\bar{C}$ no longer vanishes on-shell.

7 Discussion

In this letter we have analysed the properties of non-local gauge invariant functionals by studying some simple examples. We used general extension methods to compute gauge invariant functionals $\mathcal{O}^{\uparrow\Psi}$ by transporting a local functional $\mathcal{O}$ defined in a specified base gauge Ψ away from this gauge. We have looked explicitly at gauge invariant extensions for the mass dimension two functional $\mathcal{O} = \frac{1}{2}\int d^4x A_\mu^2$ in Maxwell's theory. From our previous analysis [7] in Yang-Mills theories it follows that these extensions have to be non local.

The non-local functionals encountered in our computation of gauge invariant extensions from general linear gauges (10) are not of the type that can be handled perturbatively. The non localities result from long range fluctuations that can not be renormalised by perturbative methods even when one calls for an infinite set of local counterterms. In this sense, the functionals in our examples are not renormalisable in the modern sense. The situation for gauge invariant extensions in Yang-Mills theories for a functional of the form $\mathcal{O} = \frac{1}{2}\int d^dx\,(A^a_\mu)^2$ is even more problematic. Besides having to deal with the same type of long range non localities the various non localities interact in a non-polynomial way.

We are well used to the idea that we need to fix the gauge in perturbation theory. However, when dealing with (perturbatively) local functionals we know that by changing the gauge all the counterterms remain local in accordance to the Quantum Action Principle. For the non-local gauge invariant extensions it is only in the base gauge that the counterterms are guaranteed to be local.

Therefore, renormalisability can only be claimed with reference to one particular gauge [4, 5, 6]. In other words, the only known way to make gauge invariant extensions renormalisable is by redefining renormalisability by *construction* in the base gauge of the extension, i.e. $\langle \mathcal{O}^{\uparrow\Psi}\rangle_{\Psi'} = \langle \mathcal{O}\rangle_{\Psi}$ for any gauge Ψ'. In this way there is a clear prescription to claim $\langle \mathcal{O}^{\uparrow\Psi}\rangle$ to be "renormalisable" – however the procedure is gauge dependent. As a result, the theory only lives in one gauge with reference to which any calculation of quantities involving insertions of $\mathcal{O}^{\uparrow\Psi}$ is possible. A well-known example of this situation is illustrated by the Curci-Ferrari model [18, 19].

This makes unclear the status of the physical relevance of $\mathcal{O}^{\uparrow\Psi}$ although it is gauge invariant. At the very least, a necessary condition for the relevance of the constructed gauge invariance of $\mathcal{O}^{\uparrow\Psi}$ is the existence of a renormalisation procedure without reference to a specific gauge.

In addition, by constructing non-local gauge invariant extensions from local functionals there is, in principle, an endless line of candidates for observables. Each can be made local in a particular "proper" gauge, as our examples illustrate. The extension procedure is too generic and does not provide *by itself*, and without the constraint of perturbative locality [7], a strong claim to support the physical relevance for a functional that it is not gauge invariant.

We conclude that a well defined meaning of such functionals without reference to the gauge where they are local and polynomial is missing. The current methods used to compute renormalised functionals require assumptions that are only known to be fulfilled by perturbatively local functionals but not by the type of non-local functionals found in the present letter. The development of the non-perturbative methods to renormalise non-local functionals in a gauge independent manner without the constraint of the Quantum Action Principle might help to improve our understanding about the relevance of gauge invariant extensions which are not perturbatively local.

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